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LETTER TO THE EDITOR

Angular momentum in electrodynamics and an argument against the existence of magnetic monopoles

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Abstract. The definition of angular momentum (AM) in electrodynamics is re-examined, so as to cover charged and infrared fields as well. The total AM is identified as that outgoing in the future or incoming from the past causal directions respectively. For these representations to be equivalent, a certain condition must be satisfied; it is fulfilled in the usual field-theory context but excludes magnetic monopoles. In the presence of charges, the total AM cannot be completely separated in remote past or future into free matter and free electromagnetic field parts, which should have consequences in quantum theory.

One of the main problems of quantum electrodynamics, which has not so far found satisfactory clarification, is the description of charged states (for a review, see [1]). On the calculational level, if the usual rules of the conventional quantum field theory are applied, it manifests itself by producing infrared divergencies. The procedures used for obtaining finite results of calculations do not tell us much about the real underlying physical situation. Analysing the QFT scheme more critically one realises that the usual Fourier representation of the angular momentum (AM) becomes meaningless even for the classical free electromagnetic field with non-vanishing infrared part, e.g. the radiation field produced by a charged particle which changes its asymptotic velocity. (This is directly related to non-absolute integrability of the AM density of the Coulomb field over any space-like hyperplane—see e.g. [2] and below.) The canonical quantization which presumes such a generator of the Lorentz transformations must lead to difficulties. In the rigourous algebraic approach to the quantum electrodynamics it is claimed that the Lorentz symmetry must be broken [1]. We do not regard this result as conclusive. The Poincaré symmetry seems to constitute a better justified demand then some of the assumptions inherent in this approach. One of them is, in effect, a classical treatment of the space-like asymptotics of electromagnetic fields (hence also of charges). We think that a consistent theory demands also the Coulomb field to be quantized. That such a line of approach can be contemplated is implied by the results of Staruszkiewicz [3].

In the present letter we indicate that in classical electrodynamics there exists an unambiguous extension of AM to the typical situation met in field theory, where massive charged particles are scattered from initial to final free asymptotic states. The consistency condition for this extension is the absence of magnetic charges; this constitutes our argument against their existence. More detailed exposition together with an attempt at quantization taking into account the results presented here will be published elsewhere.

The main procedure to be used is the asymptotic expansion of fields in null infinity (without, however, Penrose's conformal compactification of spacetime, as applied in similar

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context by Ashtekar [4]). The technical devices to be employed are the spinor calculus (in the abstract index notation) and the spin-coefficient formalism (for a complete exposition see [5]). To introduce them, let t^a be a timelike vector, x^a a point in Minkowski space and $l^a = o^A o^{A'}$ ($n^a = \iota^A \iota^{A'}$) a null future-pointing vector field orthogonal to the future (past) light-cones originating on the t^a-time-axis. o^A and ι^A form a normalized spinor basis, $o_A \iota^A = 1$; we assume that it is constant along the generators of the cones. Then $t^a = t{\cdot}n,\ l^a + t{\cdot}l,\ n^a,\ x^a = s',\ l^a + s,\ n^a$ (s and s', scaled properly, are the retarded and the advanced time of x^a, respectively). The only non-vanishing weighted spin-coefficients in the formalism based on o^A and ι^A are $\rho = -t{\cdot}l/r$ and $\rho' = t{\cdot}n/r$, where r is the radius of 2-spheres along which the retarded and advanced light cones intersect. The quantities ρ, s, o^A, $o^{A'}$ (or alternatively ρ', s', ι^A, $\iota^{A'}$) can be used as coordinates. Any weighted quantity of $\{p, q\}$-type is then a homogeneous function with the scaling law

$$f\left(\lambda\bar{\lambda}\rho, \lambda\bar{\lambda}s, \lambda o, \bar{\lambda}\bar{o}\right) = \lambda^p \bar{\lambda}^q f(\rho, s, o, \bar{o}).$$

The weighted operators of the compacted spin-coefficient formalism are then simply expressed by

$$\text{þ} f = \rho^2 \frac{\partial f}{\partial \rho} \qquad \text{þ}' f = \frac{\partial f}{\partial s} + \rho' \rho \frac{\partial f}{\partial \rho}$$
$$\eth f = -\rho\, \iota^{A'} \frac{\partial f}{\partial o^{A'}} \qquad \eth' f = -\rho\, \iota^{A} \frac{\partial f}{\partial o^{A}}.$$

Let F, in particular, be a $\{0, 0\}$-field with the future null infinity asymptotics $F(\rho, s, o, \bar{o}) = (-\rho)\Phi(s, o, \bar{o}) + \mathrm{O}\left(\rho^2\right)$ for $\rho \to 0$. Φ is then a homogeneous function of its variables with the scaling law

$$\Phi\left(\lambda\bar{\lambda}s, \lambda o, \bar{\lambda}\bar{o}\right) = (\lambda\bar{\lambda})^{-1}\Phi(s, o, \bar{o}).$$

Now the important thing is that this homogeneity property guarantees the time-axis independence of Φ and a simple transformation law with the change of the origin in Minkowski space. Namely, if $\tilde{t}^a$ is the new time-axis versor and $x^a = \tilde{s}',\ \tilde{l}^a + \tilde{s},\ \tilde{n}^a = s',\ l^a + s,\ n^a + a^a$, and $\tilde{\Phi}$ is the new asymptotic, then

$$\tilde{\Phi}\left(\tilde{s}, \tilde{o}, \bar{\tilde{o}}\right) = \Phi\left(\tilde{s} - a{\cdot}\tilde{l}, \tilde{o}, \bar{\tilde{o}}\right).$$

We apply now this asymptotic expansion to the case of electrodynamics. Let the antisymmetric field F_{ab} satisfy the generalized Maxwell equations with a possible magnetic current admitted:

$$\nabla_a F^{ab} = 4\pi J^b_{\mathrm{el}} \qquad \nabla^*_a F^{ab} = -4\pi J^b_{\mathrm{mag}}. \tag{1}$$

Both the electric and the magnetic currents are assumed to be carried by massive matter. Outside the sources the free Maxwell equations are satisfied, which in the compacted spin-coefficient formalism can be written in the form

$$\eth'\left(o^A\varphi_{AB}\right) = (\text{Þ} - \rho)\left(\iota^A\varphi_{AB}\right) \qquad \eth\left(\iota^A\varphi_{AB}\right) = \left(\text{Þ}' - \rho'\right)\left(o^A\varphi_{AB}\right)$$

where φ_{AB} is the unique symmetric spinor representing the tensor F_{ab} by $F_{ab} = \varphi_{AB}\,\epsilon_{A'B'} + \bar{\varphi}_{A'B'}\,\epsilon_{AB}$. In all cases of physical interest one can assume the future null infinity asymptotic expansion (cf [4] and [6]):

$$\varphi_{AB} = (-\rho)\,\varphi^{(1)}_{AB} + \rho^2\varphi^{(2)}_{AB} + O\left(\rho^3\right).$$

From the theory of weighted spherical harmonics [5] it follows that there exists a unique spinor $\zeta_A(s, o, \bar{o})$ such that $o^A\varphi^{(2)}_{AB} = -\rho^{-1}\eth\zeta_B = \iota^{A'}\partial\zeta_B/\partial o^{A'}$. Setting this into the equations one finds that

$$\varphi_{AB} = (-\rho)\,o_A\dot{\zeta}_B(s, o, \bar{o}) + O\left(\rho^2\right) \tag{2}$$

and

$$o_A\zeta^A = Q = Q_{\text{el}} - \mathrm{i}Q_{\text{mag}} \tag{3}$$

where the dot denotes differentiation with respect to s and Q_{el} and Q_{mag} are the electric and the magnetic charge, respectively. ζ_B is a homogeneous function with the scaling law

$$\zeta_B\left(\lambda\bar{\lambda}s, \lambda o, \bar{\lambda}\bar{o}\right) = \lambda^{-1}\zeta_B(s, o, \bar{o}). \tag{4}$$

In addition to φ_{AB} another field needed for the evaluation of angular momentum is $\varphi_{AB}\,x^B_{A'} = s'\,\varphi_{AB}o^B o_{A'} + s\,\varphi_{AB}\iota^B\iota_{A'}$. Using the identity $s'\rho + s\rho' = -1$, the definition of ζ_A, the differential identity (following from homogeneity) $\dot{\zeta}_B + o^{A'}\partial\zeta_B/\partial o^{A'} = 0$ and the asymptotic form of φ_{AB} one can show that

$$\varphi_{AB}\,x^B_{A'} = (-\rho)\,\frac{\partial}{\partial o^{A'}}\zeta_A(s, o, \bar{o}) + O\left(\rho^2\right) = (-\rho)\,o_A\,\nu_{A'}(s, o, \bar{o}) + O\left(\rho^2\right). \tag{5}$$

The last equality in (5) is a consequence of (3); it defines the spinor $\nu_{A'}$. We observe that the leading asymptotic terms of φ_{AB} and $\varphi_{AB}\,x^B_{A'}$ are expressed by a single homogeneous spinor function. Conversely, if these fields have the assumed asymptotic form, then this form uniquely determines the spinor ζ_A (this again follows from the theory of weighted spherical harmonics).

We assume now further that similar asymptotic expansions exist in the past null infinity. The coordinate system appropriate for this region is $\left(\rho', s', \iota^A, \iota^{A'}\right)$ and the asymptotic formulae are obtained from (2)–(5) by substituting $\left(\rho', s', \iota^A, \iota^{A'}\right)$, ζ'_B and $\nu'_{A'}$ for $\left(-\rho, s, o^A, o^{A'}\right)$, ζ_B and $\nu_{A'}$ respectively.

Finally, we restrict the class of admisible currents: the radiation field produced by them (the retarded minus the advanced field) should belong to the class of fields of the assumed type. Among the currents thus admitted are those carried by massive charged particles or massive charged fields (e.g. Klein–Gordon or Dirac fields) evolving freely for past and future asymptotic times.

The following question remains to be answered: what are the conditions for ζ_B and ζ'_B (subject to (3) and (4)) to represent the actual asymptotics of some existing field with the currents in the assumed class. We shall not analyse this question here in detail but only state the results. The following demands comprise a necessary condition: there exist limits of $\zeta_B(s,o,\bar{o})$ and $\zeta'_B(s',\iota,\bar{\iota})$ for s and s' tending to plus and minus infinity and $\zeta'_B(+\infty,\iota,\bar{\iota}) = \zeta_B(-\infty,\iota,\bar{\iota})$. The addition of some further conditions on the rates at which all those limits are attained transforms the condition into a sufficient one. The rate estimates which can hardly be weakened and which are assumed consequently are the following: $\dot{\zeta}_B(s,o,\bar{o})$ together with its 1. and 2. spinor derivatives is $O(s^{-1-\epsilon})$ for $|s| \to \infty$ and similarly for $\dot{\zeta}'_B$. Moreover, in the case of vanishing sources one has $\zeta_B(s,o,\bar{o}) + \zeta'_B(s,o,\bar{o}) = \zeta_B(-\infty,o,\bar{o}) = \zeta'_B(+\infty,o,\bar{o})$ (hence $\zeta_B(+\infty,o,\bar{o}) = \zeta'_B(-\infty,o,\bar{o}) = 0$); cf [6].

The fields φ_{AB} and $\varphi_{AB}\, x^B_{A'}$ thus determined have definite asymptotic limits in space-like directions as well. For $r \to \infty$, the fields in any hyperplane orthogonal to the time-axis are of order

$$\varphi_{AB} = O\left(r^{-2}\right) \qquad \varphi_{AB}\, x^B_{A'} = O\left(r^{-1}\right) \tag{6}$$

and the leading terms are completely expressible in terms of $\zeta_B(-\infty,o,\bar{o})$, although not as simply as in the null case. If $\zeta_B(-\infty,o,\bar{o}) = 0$, the leading terms vanish.

Now we can turn to the evaluation of the AM. We assume that the equations (1) form a part of a closed theory with a locally conserved symmetric energy–momentum tensor T_{ab}. Outside the matter T_{ab} reduces to the usual electrodynamical tensor, $T_{ab} = (2\pi)^{-1}\, \bar{\varphi}_{A'B'}\, \varphi_{AB}$ in spinor language, which yields the angular momentum density

$$x_a T_{bc} - x_b T_{ac} = -\frac{1}{2\pi}\, \epsilon_{A'B'}\, \bar{\varphi}_{C'D'}\, x^{D'}_{(A}\, \varphi_{B)C} - \frac{1}{2\pi}\, \epsilon_{AB}\, \varphi_{CD}\, x^{D}_{(A'}\, \bar{\varphi}_{B')C'}\,.$$

If $\zeta_B(-\infty,o,\bar{o}) \neq 0$ then this expression is not absolutely integrable on any space-like hyperplane and the total AM cannot be straightforwardly defined (this fact was clearly stated for the Coulomb field in [2]). The remarkable thing is, however, that the amount of AM radiated into any solid angle during any time lapse, as well as that contained in any light-cone, is finite. To see, this let us fix the time axis and choose the scaling of spinors so that $t \cdot l = 1$; s is then the retarded time. Moreover, for any AM tensor M_{ab} we introduce a corresponding AM spinor μ_{AB} by the standard decomposition $M_{ab} = \mu_{AB}\, \epsilon_{A'B'} + \bar{\mu}_{A'B'}\, \epsilon_{AB}$. Using (2) and (5) one shows that the spinor of the AM radiated into the null infinity is

$$-\frac{1}{2\pi} \int \bar{\nu}_{(A} \dot{\zeta}_{B)}\, (s,o,\bar{o})\, \mathrm{d}s\, \mathrm{d}\Omega$$

where $\mathrm{d}\Omega$ is the angle measure and the inegration extends over the proper solid angle and retarded time span. The fall off rate of the integrand guarantees its absolute integrability over any retarded time interval. The integrability of the AM over any light-cone follows from the null asymptotics.

With the above results in mind, let us concentrate on the situation depicted in figure 1. $C(s_i)$ are the future light-cones originating from points on the time-axis and Σ is a hyperplane orthogonal to the axis and cutting it between s_1 and s_2. $\Sigma(s_1)$ is the compact

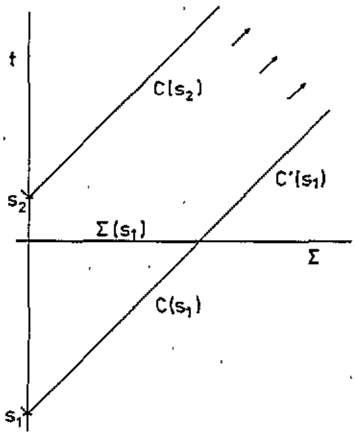

Figure 1. The whole Minkowski space is represented by the rotation of the figure around the time-axis t.

portion of Σ cut off by the cone $C(s_1)$; conversely, $C'(s_1)$ is the unbounded portion of $C(s_1)$ cut off by Σ. The local conservation law and its inegrability imply

$$\mu_{AB}^{C(s_1)} = \mu_{AB}^{C(s_2)} + \mu_{AB}^{\mathrm{rad}(s_1,s_2)} = \mu_{AB}^{\Sigma(s_1)} + \mu_{AB}^{C'(s_1)} \tag{7}$$

where $\mu_{AB}^{\mathrm{rad}(s_1,s_2)}$ is the spinor of the AM radiated into the null infinity between $C(s_1)$ and $C(s_2)$ and other spinors represent the AM contained in the surface indicated in the superscript. $\mu_{AB}^{C(s)}$ is an instant of what in the gravitation theory is called a Bondi quantity. The first equality in (6) and the formula for the radiated AM show that the finite limits $\mu_{AB}^{C(+\infty)}$, $\mu_{AB}^{C(-\infty)}$ and $\mu_{AB}^{\mathrm{rad}(-\infty,+\infty)}$ exist. Moreover, it follows from the large $|s|$ behaviour of ζ_B that all these quantities are independent of the time-axis versor choice. For the total radiated AM this is also seen by the explicit formula, in which the special scaling of spinors (which breaks the invariance) need not to be assumed:

$$\mu_{AB}^{\mathrm{rad}} = -\frac{1}{2\pi}\int \bar{\nu}_{(A}\dot{\zeta}_{B)}\,(s, o, \bar{o})\,\mathrm{d}s\,\mathrm{d}^2 l. \tag{8}$$

Here $\mathrm{d}^2 l$ is the invariant homogeneous measure over the null directions l^a with the scaling law $\mathrm{d}^2 l\left(\lambda o, \bar{\lambda}\bar{o}\right) = \left(\lambda\bar{\lambda}\right)^2 \mathrm{d}^2 l\,(o, \bar{o})$; its explicit form given in [5] is $\mathrm{d}^2 l = \iota o_{A'}\,\mathrm{d}o^{A'} \wedge o_A\,\mathrm{d}o^A$; see also [7]. For the whole integral to be invariant the integrand of $\mathrm{d}^2 l$ should scale with $\left(\lambda\bar{\lambda}\right)^{-2}$, which indeed is the case.

The time-axis independence admits the clear-cut interpretation of $\mu_{AB}^{C(+\infty)}$ as the AM spinor of the outgoing matter, with its Coulomb field, after all radiation has died out. The total AM going out in the future causal (null and time-like) directions is $\mu_{AB}^{C(-\infty)} = \mu_{AB}^{C(+\infty)} + \mu_{AB}^{\mathrm{rad}}$. It would be natural to identify this quantity as the total AM of the system. For this identification to be admissible, however, a certain physical consistency condition should be satisfied. To see this we take into account the second equality in (7). One of the results of the reported work, which we shall not prove here, is that there exists Lorentz invariant (time-axis versor independent) limit of the second term on the RHS of (7) given by

$$\mu_{AB}^{C'(-\infty)} = \frac{1}{4\pi}\int \bar{\nu}_{(A}\zeta_{B)}\,(-\infty, o, \bar{o})\;\mathrm{d}^2 l. \tag{9}$$

Consequently, a finite limit $\mu_{AB}^{\Sigma(-\infty)}$ of the first term on the RHS of (7) also exists. (How is that possible if the asymptotic expansion (6) holds? The answer is that the leading term of the integrand changes sign with the 3-space Σ reflection—a property reminiscent of analogous conditions formulated in the gravitation theory by Regge and Teitelboim [8].) $\mu_{AB}^{\Sigma(-\infty)}$ is what in the absolutely integrable case would be the total AM spinor; we see that it differs, however, from the total outgoing AM spinor by $\mu_{AB}^{C'(-\infty)}$. This fact alone would not constitute a fatal difficulty as $\mu_{AB}^{\Sigma(-\infty)}$ is a result of some not completely unambiguous regularization. However, one can repeat our whole construction with the time direction reversed, that is with the cones directed into the past. It turns out then that the spinor of the difference of the total AM going out in the future causal directions and the total AM incoming from the past causal directions is given by twice the expression (9). If $\mu_{AB}^{C'(-\infty)} \neq 0$ then the AM leaks out into the space-like infinity; an unambiguous identification of the total AM is not possible in that case. We demand therefore that (9) vanishes. Now the integrand of (9) containes the information on the asymptotic currents for time tending to plus and minus infinity; if both electric and magnetic charges are present the integral does not in general vanish. This is our argument against the existence of the latter. A general sufficient condition for vanishing of (9) is that, loosely speaking, the infrared and charged sectors are only of electric (but not magnetic) type. In precise mathematical terms: from the homogeneity property of $\zeta_A(-\infty, o, \bar{o})$ one has $\nu_{A'}(-\infty, o, \bar{o}) = o_{A'}\nu(o, \bar{o})$ (which also defines $\nu(o, \bar{o})$. Our sufficient condition is

$$\bar{\nu}(o, \bar{o}) = \nu(o, \bar{o})\,.$$

General asymptotic currents carried by freely moving electrically charged massive particles or fields satisfy the condition.

If our criterion is satisfied, we obtain a well-founded identification of the total AM spinor

$$\mu_{AB}^{\mathrm{tot}} = \mu_{AB}^{\mathrm{out}} + \mu_{AB}^{\mathrm{rad}} \tag{10}$$

where $\mu_{AB}^{\mathrm{out}} \equiv \mu_{AB}^{C(+\infty)}$ is the AM spinor of the outgoing matter and μ_{AB}^{rad}, given by (8), is the radiated AM spinor; similar decomposition holds with respect to the past. We note that in the presence of charges the radiated AM cannot be due solely to any free radiation field; we shall not elaborate this point here but note only that in that case $o_A\zeta^A \neq 0$, contrary to what the free field case would yield. (At this point one should comment on the apparent contradiction of our result with those of Lozada and Torres [2]. Those authors rightly observe that the AM density is in general non-absolutely-integrable and various choices of improper limit give different results: the authors produce explicitely two special limits on a particular hyperplane for a special case of a point-charge field. They claim that there are no physical reasons to choose a particular way to reach the spatial infinity. The point is, however, that our approach *does* furnish strong support for our definition, which is the same as taking the limit $\lim_{s\to-\infty}\mu_{AB}^{C(s)}$. As discussed above this definition is time-axis independent and covers all situations of physical interest in field theory.)

Finally, we mention for completeness that the difficulties met for AM are absent in the energy–momentum case. The expression analogous to (10) is then

$$P_a^{\text{tot}} = P_a^{\text{out}} + P_a^{\text{rad}}$$

with

$$P_a^{\text{rad}} = \frac{1}{2\pi} \int \bar{\dot{\xi}}_{A'} \dot{\xi}_A (s, o, \bar{o}) \, \mathrm{d}s \, \mathrm{d}^2 l \, .$$

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References

[1] Morchio G and Strocchi F 1986 *Fundamental Problems of Gauge Field Theory* ed G Velo and A S Wightman (New York: Plenum)

[2] Lozada A and Torres P L 1988 *J. Math. Phys.* **29** 1361

[3] Staruszkiewicz A 1989 *Ann. Phys., NY* **190** 354

[4] Ashtekar A 1986 *Asymptotic Quantization* (Bibliopolis)

[5] Penrose R and Rindler W 1984 *Spinors and Spacetime* vol 1 (Cambridge: Cambridge University Press)

[6] Chakravarty S, Ivancovich J and Newman E T 1986 *Gen. Rel. Grav.* **18** 633

[7] Zwanziger D 1976 *Phys. Rev.* D **14** 2570

[8] Regge T and Teitelboim C 1974 *Ann. Phys., NY* **88** 286